MATH5360 Game Theory
Exercise 1

Assignment 1: 1,2(a)(c)(e)(g),3,5,7,11,13 (Due: 9 March 2020 (Monday))

1. Find the values of the following game matrices by finding their saddle points

(a) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -4 & 5 & -3 & -3 \\ 0 & 1 & 3 & -1 \\ -3 & -1 & 2 & -5 \\ 2 & -4 & 0 & -2 \end{pmatrix}$

2. Solve the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column.

(a) $\begin{pmatrix} 1 & 7 \\ 2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 2 & -3 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & -3 \\ -3 & 5 \\ 2 & -1 \\ 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -2 & 4 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$

3. Raymond holds a black 2 and a red 9. Calvin holds a red 3 and a black 8. Each of them chooses one of the cards from his hand and then two players show the chosen cards simultaneously. If the chosen cards are of the same colour, Raymond wins and Calvin wins if the cards are of different colours. The loser pays the winner an amount equal to the number on the winner's card. Write down the game matrix, find the value of the game and the optimal strategies of the players.

4. Alex and Becky point fingers to each other, with either one finger or two fingers. If they match with one finger, Becky pays Alex 3 dollars. If they match with two fingers, Becky pays Alex 11 dollars. If they don't match, Alex pays Becky 1 dollar.

   (a) Find the optimal strategies for Alex and Becky.

   (b) Suppose Alex pays Becky k dollars as a compensation before the game. Find the value of k to make the game fair.

5. Player I and II choose integers i and j respectively where $1 \leq i, j \leq 7$. Player II pays Player I one dollar if $|i - j| = 1$. Otherwise there is no payoff. Write down the game matrix of the game, find the value of the game and the optimal strategies for the players.

6. Use the principle of indifference to solve the game with game matrix

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. In the Mendelsohn game, two players choose an integer from 1 to 5 simultaneously. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1 dollar from its opponent. The player that chooses a number two or more larger than his opponent loses 2 dollars to its opponent. Find the game matrix and solve the game.

8. Aaron puts a chip in either his left hand or right hand. Ben guesses where the chip is. If Ben guesses the left hand, he receives \$2 from Aaron if he is correct and pays \$4 to Aaron if he is wrong. If Ben guesses the right hand, he receives \$1 from Aaron if he is correct and pays \$3 to Aaron if he is wrong.

   (a) Write down the payoff matrix of Aaron. (Use order of strategies: Left, Right.)

   (b) Find the maximin strategy for Aaron, the minimax strategy for Ben and the value of the game.

9. Let

$$A = \begin{pmatrix} -3 & 1 \\ c & -2 \end{pmatrix}$$

   where c is a real number.

   (a) Find the range of values of c such that A has a saddle point.

   (b) Suppose the zero sum game with game matrix A is a fair game.

      (i) Find the value of c.

      (ii) Find the maximin strategy for the row player and the minimax strategy for the column player.

10. Prove that if A is a skewed symmetric matrix, that is, $A^T = -A$, then the value of A is zero.

11. Let n be a positive integer and $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbb{R}^n$. Prove the following statements.

   (a) If A is an $n \times n$ symmetric matrix, that is $A^T = A$, and there exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T = v\mathbf{1}^T$ where $v \in \mathbb{R}$ is a real number, then v is the value of A.

   (b) There exists an $n \times n$ matrix A, a probability vector $\mathbf{y} \in \mathcal{P}^n$ and a real number v such that $A\mathbf{y}^T = v\mathbf{1}^T$ but v is not the value of A.

12. Let n be a positive integer and

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

be an $n \times n$ diagonal matrix where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(a) Suppose $\lambda_1 \leq 0$ and $\lambda_n > 0$. Find the value of the zero sum game with game matrix D.

(b) Suppose $\lambda_1 > 0$. Solve the zero sum game with game matrix D.

13. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find a vector $\mathbf{x} = (1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ and a real number a such that

$$A\mathbf{x}^T = (0, 0, 0, 0, a)^T$$

(b) Find a vector $\mathbf{y} = (1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$ and a real number b such that

$$A\mathbf{y}^T = (1, 1, 1, 1, b)^T$$

(c) Find the maximin strategy, the minimax strategy and the value of A. (Hint: Find real numbers $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{q} = \alpha\mathbf{x} + \beta\mathbf{y}$ satisfies $A\mathbf{q}^T = v\mathbf{1}^T$ for some $v \in \mathbb{R}$.)

14. For positive integer k, define

$$A_k = \begin{pmatrix} 4k-3 & -(4k-2) \\ -(4k-1) & 4k \end{pmatrix}.$$

(a) Solve A_k, that is, find the maximin strategy, minimax strategy and value of A_k in terms of k.

(b) Let $r_1, r_2, \cdots, r_n > 0$ be positive real numbers. Using the principle of indifference, or otherwise, find, in terms of $r_1, r_2, \cdots, r_n$, the value of

$$D = \begin{pmatrix} \frac{1}{r_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{r_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{r_n} \end{pmatrix}.$$

(c) Find, with proof, the value of the matrix

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{25} \end{pmatrix}.$$